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DYNAMICAL TORSION OF VISCOELASTIC CONE

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ABSTRACT. Propagation of torsion waves in a viscoelastic cone of semi-infinite and finite length are examined by the Laplace integral transform method and the method of separation of variables. The inverse transforms for elastic cone are obtained using the contour integration method and the residue theory. Using this solution and the solution of auxiliary one-dimensional problem for viscoelastic half-space with any hereditary function, it is proved that the solution for viscoelastic cone is in the form of generalized convolution of these solutions. The solution of the auxiliary problem for any creep kernel is constructed in the form of absolutely and uniformly convergent series. As an example the solution of the auxiliary problem has been obtained for the creep kernel as the sum of exponential functions, for Maxwell and the standard linear solid models and for the weakly singular Abel kernel. Solutions for dynamical problems of viscoelastic half-space under the torsion point action, half-space with the spherical cavity on the surface and circular cylinder are obtained as particular cases.

Keywords: viscoelastic cone, creep kernel, half-space, cylinder, Bessel and Legendre functions, Laplace transform.

AMS Subject Classification: 74H05; 74J05.

1. Introduction

It is known that the torsion waves in a circular cylindrical bar do not have dispersion. This makes possible to obtain exact solutions of axially symmetric problems on propagation of torsion waves in elastic and viscoelastic cylinders [8, 17, 18, 23, 24]. Solution of the problem is reduced to the eigen-value problems and represented by the sum of eigen-functions, each of which corresponds to some of free vibrations connecting their frequencies. The same situation is also correct for elastic conical bars if the ends of the cone are the coordinate surfaces in a spherical coordinate system [3, 16, 22]. In [22] the solution for the elastic cone is obtained for small and large values of time and the elementary solution, for which each spherical section remains spherical and turns around the axis of the cone, is investigated in details. The exact solutions for elastic cone in the form of the series of eigen-functions are obtained in [3, 16].

The exact solution in the series of eigen-functions of the problem of propagation of torsion waves in viscoelastic cones of finite and infinite length are obtained in this paper. The viscoelastic constitutive equations are written in the Boltzmann-Volterra form. The problems are solved by the Laplace integral transform methods and the method of separation of variables. The inverse Laplace transforms for an elastic cone are found by the contour integration method and the residue theory. It is proved that the solution for a viscoelastic cone is in the form of generalized convolution of corresponding solution for an elastic cone with the solution of auxiliary problem which coincides with the one-dimensional problem of propagation of transient shear waves in the viscoelastic half-space for any hereditary function. This problem is also solved by using the

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Laplace transform. For any hereditary function the contour integral method is not available here for obtaining the inverse Laplace transform, thus the analytic expression of integrand is not given explicitly. For using this method it is necessary to know the poles and the branch points of integrand in the contour integral, which is equivalent to specify analytic expression of material functions. The simulation of viscoelastic property of material by means of specific functions impairs the generality of description of the behavior of real materials and imposes restrictions that do not at all follow from the fundamental laws of nature. Such approaches are considered in [1, 7, 10, 19]. Review of the early literature on the subject is given in [10], [14]. The Rayleigh problem for viscoelastic solids and fluids is considered in [14] and the Bromwich integral is reduced to the generalized integral which in a complicated way depends on real and imaginary parts of the relaxation function.

For the real viscoelastic materials we need to have an explicit solution for any hereditary function useful for theoretical and practical applications. A new solution technique which completely excludes the above mentioned difficulties and makes possible to construct the solution for any creep kernel$G(t)$ is presented in the paper. The method is based on the expansion of the image of searching function to the absolutely and uniformly convergent series on the powers of the Laplace transform of the creep kernel $\bar{G}(s)$,where s is the parameter of the Laplace transform. Original of the series is obtained by term by term calculation of the inverse Laplace transforms. As an example the regular kernel, the kernel of the sum of exponential functions, the weakly singular kernel and the Maxwell model are considered. The convergence of the series in obtained solution is carried out. The behavior of the solution of auxiliary problem on the wave front propagating by the equilibrium speed is also investigated. The solutions for the half-sphere and half-space with a spherical cavity on the surface, for the half-space and for the circular cylinder are obtained as particular cases.

2. Statement of the problem

Let us consider the propagation of axially symmetric non-stationary torsion waves in the viscoelastic cone. The cone is in its natural state and occupies the domain $D = \{r_1 < r < r_2,\ 0 \le \theta < \theta_0,\ 0 \le \varphi < 2\pi\}$ in the spherical coordinates $r,\ \theta,\ \varphi$. Body forces and surface tractions vanish. The problem is reduced to the solution of the integro-differential equation [2, 10, 13, 20]

$$\mu^* L u \equiv \mu^* \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{u}{r^2 \sin^2\theta} \right) = \rho \frac{\partial^2 u}{\partial t^2}, \quad r, \theta \in D, t > 0 \qquad (1)$$

for the initial and boundary conditions below

$$u = \frac{\partial u}{\partial t} = 0, \quad t = 0, \quad r, \theta \in D, \qquad (2)$$

$$\sigma_{\theta\varphi} = 0, \quad \theta = \theta_0, \quad r \in (r_1, r_2), t > 0, \qquad (3)$$

$$u = a_1(\theta, t), \quad r = r_1; \quad u = a_2(\theta, t), \quad r = r_2, \quad \theta \in [0, \theta_0], t > 0, \qquad (4)$$

where $u(r, \theta, t)$ is the displacement, ρ is density, $a_i(\theta, t)$ are the given continuous functions on $0 \le \theta \le \theta_0,\ t \ge 0$, satisfying the conditions $a_i(0, t) = 0$,

$$\sigma_{\theta\varphi} = \mu^* \left[\frac{1}{r} \left(\frac{\partial u}{\partial \theta} - u \cot\theta \right) \right], ...$$

are the components of stress tensor, the asterisk denotes the integral operator of the form

$$\mu^* f(t) = \int_0^t \mu(t-\tau)\, df(\tau),$$

where $\mu(t)$ is the shear relaxation function. It is known that $\mu(t)$ is monotonically decreasing continuous function, having monotonically increasing integrable derivative on $t > 0$, moreover, $0 < \mu(\infty) = \mu_\infty \le \mu(t) \le \mu(0) = \mu_0$ where μ_∞ and μ_0 are the equilibrium and instantaneous shear elastic module, respectively and $\mu(t) \equiv 0$ for $t < 0$.

For the uniqueness, while evaluating the integrals we will suppose that

$$\int_0^t f(\tau)\, d\tau \equiv \lim_{\alpha\to 0} \int_{-\alpha}^{t+\alpha} f(\tau)\, d\tau = \int_{0^-}^{t^+} f(\tau)\, d\tau,$$

where α is positive.

3. Solution of the problem by the Laplace transform

The Laplace transform of the function $f(t)$ is the function $\bar{f}(s)$ of the parameter s defined for all complex values of this parameter by the relation

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

where the integral is evaluated for the positive semi axis. It is assumed that the integral on the right-hand side always exists. For this, the function$f(t)$ must be integrable everywhere, can not grow faster than exponential function when t tends to infinity and for non-positive values of the argument it must be identically zero [9, 12].

Applying the Laplace transform for t to (1), taking (2) into account and replacing $\gamma = \cos\theta$ we find the equation

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \gamma}\left[\left(1-\gamma^2\right)\frac{\partial \bar{u}}{\partial \gamma}\right] - \frac{\bar{u}}{r^2\left(1-\gamma^2\right)} - \eta^2 \bar{u} = 0,$$

where $\eta^2 = \rho s^2 / (s\bar{\mu})$ and the bar above the function denotes its Laplace transform with the parameter s. Seeking for the solution of the obtained equation by separation of variables as $\bar{u}(r,\gamma) = R(r)\, X(\gamma)$, gives

$$R^{-1}\left[r^2\left(R'' + \frac{2}{r}R' - \eta^2 R\right)\right] = -X^{-1}\left\{\frac{d}{d\gamma}\left[\left(1-\gamma^2\right)X'\right] - \frac{1}{1-\gamma^2}X\right\}.$$

The left hand side of this equation depends only of the variable r and the right hand side depends only of the variable γ. Therefore both of sides are equal to the same constant, which for conveniences we take as $\alpha(\alpha+1)$, where α will be defined below. So for the functions R and X we find

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \left[\eta^2 + \frac{\alpha(\alpha+1)}{r^2}\right]R = 0, r_1 < r < r_2, \tag{5}$$

$$\frac{d}{d\gamma}\left[\left(1-\gamma^2\right)\frac{d}{d\gamma}X\right] + \left[\alpha(\alpha+1) - \frac{1}{1-\gamma^2}\right]X = 0, \quad |X(1)| < \infty, \gamma_0 = \cos\theta_0 < \gamma < 1. \tag{6}$$

The general solution of equation (5) is

$$R(r) = \frac{1}{\sqrt{r}}\left[c_1 J_{\alpha+1/2}(i\eta r) + c_2 Y_{\alpha+1/2}(i\eta r)\right],$$

where J_α and Y_α are the Bessel functions of the first and second kinds, respectively, and c_1, c_2 are the arbitrary constants [4]. The solution of the Legendre equation (6) is written in the form

$$X(\gamma) = c_3 P_\alpha^1(\gamma) + c_4 Q_\alpha^1(\gamma), \tag{7}$$

where P_α^1 and Q_α^1 are the Legendre functions of the first and second kinds, respectively and c_3, c_4 are arbitrary constants.

For the bounded ness of the solution on the axis $\gamma = 1$ we should take $c_4 = 0$. Then putting $c_3 = 1$ we will get the image $\bar{u}$ of the solution being searched depending on three unknown parameters α, c_1 and c_2

$$\bar{u} = \frac{1}{\sqrt{r}} \left[c_1 J_{\alpha+1/2}(ir\eta) + c_2 Y_{\alpha+1/2}(ir\eta)\right] P_\alpha^1(\gamma).$$

The boundary condition (3) will be satisfied if we write

$$P_\alpha^2(\gamma_0) = 0. \tag{8}$$

The equation has infinite number of simple real roots α_n. For the small angle θ_0 we have $\alpha_n \approx \frac{1}{2} + l_n \left(2\sin\frac{\theta_0}{2}\right)^{-1}$, where l_n is the non zero root of the equation $J_1(l) = 0$. The function $\bar{u}$ is represented as the sum

$$\bar{u} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{r}} \left[c_{1n} J_{\alpha_n+1/2}(ir\eta) + c_{2n} Y_{\alpha_n+1/2}(ir\eta)\right] P_{\alpha_n}^1(\gamma). \tag{9}$$

Assuming the functions $\bar{a}_i(\gamma, s)$ are twice continuous differentiable with respect to γ in the interval $(\gamma_0, 1)$, we expand them to the uniformly and absolutely convergent series on the Legendre functions [4]

$$\bar{a}_i(\gamma, s) = \sum_{n=1}^{\infty} \bar{a}_{in}(s) P_{\alpha_n}^1(\gamma),$$

where

$$\bar{a}_{in}(s) = \frac{1}{b_n} \int_{x_0}^{1} \bar{a}_i(\gamma, s) P_{\alpha_n}^1(\gamma)\, d\gamma, \quad b_n = \int_{\gamma_0}^{1} \left[P_{\alpha_n}^1(\gamma)\right]^2 d\gamma. \tag{10}$$

From the conditions (4) we find the system of the two linear non-homogeneous algebraic equations for the coefficients c_{1n} and c_{2n}

$$c_{1n} J_{\alpha_n+1/2}(ir_1\eta) + c_{2n} Y_{\alpha_n+1/2}(ir_1\eta) = \bar{a}_{1n}\sqrt{r_1},$$

$$c_{1n} J_{\alpha_n+1/2}(ir_2\eta) + c_{2n} Y_{\alpha_n+1/2}(ir_2\eta) = \bar{a}_{2n}\sqrt{r_2}.$$

Substituting the solution of the system into (9), we get

$$\bar{u} = \sum_{n=1}^{\infty} P_{\alpha_n}^1(\gamma) \left[\sqrt{\frac{r_1}{r}} \bar{a}_{1n} \bar{\Omega}_{1n}(r, s) + \sqrt{\frac{r_2}{r}} \bar{a}_{2n} \bar{\Omega}_{2n}(r, s)\right], \tag{11}$$

where

$$\bar{\Omega}_{1n} = \frac{J_{\alpha_n+1/2}(ir\eta) Y_{\alpha_n+1/2}(ir_2\eta) - Y_{\alpha_n+1/2}(ir\eta) J_{\alpha_n+1/2}(ir_2\eta)}{J_{\alpha_n+1/2}(ir_1\eta) Y_{\alpha_n+1/2}(ir_2\eta) - Y_{\alpha_n+1/2}(ir_1\eta) J_{\alpha_n+1/2}(ir_2\eta)}$$

and $\bar{\Omega}_{2n}$ is obtained from $\bar{\Omega}_{1n}$ by replacing $r_1 \leftrightarrow r_2$.

The function $\bar{u}$ represented by (11) and the image of the relaxation function $\bar{\mu}$are analytic in the right half-plane Re$s > 0$. We suppose that these functions are analytically continued into the whole complex plane, except for some isolated points of singularity.

As we see the Laplace transform of the solution of dynamic problems of viscoelasticity irrationally depends on the Laplace transform of relaxation function, unlike the quasi-static problems. This is the main difficulty in solving the problems. We will seek for the way of removing this difficulty.

For the elastic cone, for which $\eta = s/c$ and $c = \sqrt{\mu_0/\rho}$ is the wave speed, the function $\bar{\Omega}_{1n}$ is a single-valued function of s and has simple poles situated on the imaginary axis. The inverse Laplace transform of the function $\bar{\Omega}_{1n}$ is evaluated by the Bromwich integral

$$\Omega_{1n}(r,t) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} e^{st}\bar{\Omega}_{1n}(r,s)\,ds \quad (p>0),$$

where the integration is carried out in the plane of the complex variable s along an infinite straight line parallel to the imaginary axis and situated so that all singular points of the function $\bar{\Omega}_{1n}$ are located to the left of the straight line. Using the contour integration method and residue theory we find [9]

$$\Omega_{1n}(r,t) = \left(\frac{r_1}{r}\right)^{\alpha_n+1/2}\frac{r_2^{2\alpha_n+1} - r^{2\alpha_n+1}}{r_2^{2\alpha_n+1} - r_1^{2\alpha_n+1}}\delta(t) -$$

$$-\sum_{k=1}^{\infty}\frac{\pi\lambda_{kn}cJ_{\alpha_n+1/2}(r_1\lambda_{kn})\,J_{\alpha_n+1/2}(r_2\lambda_{kn})\,B_{kn}(r)}{J^2_{\alpha_n+1/2}(r_1\lambda_{kn}) - J^2_{\alpha_n+1/2}(r_2\lambda_{kn})}\sin(\lambda_{kn}ct), \tag{12}$$

where

$$B_{kn}(r) = J_{\alpha_n+1/2}(r\lambda_{kn})\,Y_{\alpha_n+1/2}(r_2\lambda_{kn}) - J_{\alpha_n+1/2}(r_2\lambda_{kn})\,Y_{\alpha_n+1/2}(r\lambda_{kn}).$$

Here $\delta(t)$ is the Dirac delta function and λ_{kn} are the roots of the equation (all real and simple)

$$J_{\alpha_n+1/2}(r_1\lambda)\,Y_{\alpha_n+1/2}(r_2\lambda) - J_{\alpha_n+1/2}(r_2\lambda)\,Y_{\alpha_n+1/2}(r_1\lambda) = 0.$$

Using (12) and the corresponding expression for $\Omega_{2n}(r,t)$ we find the solution of the considering problem for an elastic cone, which we denote by $v(r,\theta,t)$:

$$v(r,\theta,t) = \sum_{n=1}^{\infty}P^1_{\alpha_n}(\gamma)\left[\sqrt{\frac{r_1}{r}}a_{1n}(t)*\Omega_{1n}(r,t) + \sqrt{\frac{r_2}{r}}a_{2n}(t)*\Omega_{2n}(r,t)\right], \tag{13}$$

where the asterisk between the functions denotes their convolution as

$$f(t)*g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

For obtaining the solution of problem (1)-(4) we will use solution (13) for the functions $a_i(\theta,t) = a_i(\theta)\,\delta(t) \quad (i=1,2)$. In this case function (13) is written as

$$v(r,\theta,t) = \sum_{n=1}^{\infty}P^1_{\alpha_n}(\gamma)\left[\sqrt{\frac{r_1}{r}}a_{1n}\Omega_{1n}(r,t) + \sqrt{\frac{r_2}{r}}a_{2n}\Omega_{2n}(r,t)\right], \tag{14}$$

where a_{in} are constant. It is evident that function (14) is the solution of the problem below

$$\mu_0 Lv = \rho\frac{\partial^2 v}{\partial t^2}, \quad r,\theta\in D, t>0, \tag{15}$$

$$v = \frac{\partial v}{\partial t} = 0, \quad t = 0, \quad r, \theta \in D, \tag{16}$$

$$\frac{\partial v}{\partial \theta} - v \cot \theta = 0, \quad \theta = \theta_0, \quad r \in (r_1, r_2), t > 0, \tag{17}$$

$$v = a_1(\theta)\,\delta(t), \quad r = r_1; \quad v = a_2(\theta)\,\delta(t), \quad r = r_2, \quad 0 \le \theta < \theta_0. \tag{18}$$

Theorem 3.1. *Let $v(r, \theta, t)$ be the solution of the problem* (15)-(18) *and $W(t; \tau)$ be the solution of the following auxiliary problem*

$$\frac{\mu^*}{\mu_0} \frac{\partial^2 W}{\partial \tau^2} = \frac{\partial^2 W}{\partial t^2}, \quad t > 0, \tau > 0, \tag{19}$$

$$W(0; \tau) = 0, \quad \frac{\partial W(0; \tau)}{\partial t} = 0, \quad \tau > 0, \tag{20}$$

$$W(t; 0) = H(t), \quad t > 0, \tag{21}$$

$$W(t; \tau) \to 0, \quad \tau \to \infty, \quad t > 0, \tag{22}$$

where $H(t)$ is the Heaviside step function.
Then the solution of the problem (1)-(4) *for a viscoelastic cone inthe case $a_i(\theta, t) = a_i(\theta) H(t)$ $(i = 1, 2)$ is written as*

$$u(r, \theta, t) = \int_0^t W(t; \tau)\, v(r, \theta, \tau)\, d\tau. \tag{23}$$

Proof. Putting (23) into (19), assuming the possibility of differentiation under the integral sign and taking (21) into account, we obtain

$$\mu^* \int_0^t W(t; \tau)\, Lv(r, \theta, \tau)\, d\tau = \rho \int_0^t \frac{\partial^2 W(t; \tau)}{\partial t^2} v(r, \theta, \tau)\, d\tau.$$

Since the function v satisfies equation (15), the preceding equation can be written in the form

$$\frac{\mu^*}{\mu_0} \int_0^t W(t; \tau) \frac{\partial^2 v(r, \theta, \tau)}{\partial \tau^2} d\tau = \int_0^t \frac{\partial^2 W(t; \tau)}{\partial t^2} v(r, \theta, \tau)\, d\tau.$$

By integrating by part, considering conditions (16) and (20) and then changing the order of the integrals, we find

$$\int_0^t \left[\frac{1}{\mu_0} \int_\tau^t \mu(t - \varsigma) \frac{\partial^3 W(\varsigma; \tau)}{\partial \tau^2 \partial \varsigma} d\varsigma - \frac{\partial^2 W(t; \tau)}{\partial t^2} \right] v(r, \theta, \tau)\, d\tau = 0.$$

On the basis of (19), this equation is satisfied identically. The initial condition (2) is satisfied on the basis of conditions (16) and (20). Using (18) and (21), we find

$$u(r_i, \theta, t) = \int_0^t W(t; \tau)\, v(r_i, \theta, \tau)\, d\tau = \int_0^t W(t; \tau)\, a_i(\theta)\, \delta(\tau)\, d\tau = a_i(\theta)\, W(t; 0) = a_i(\theta)\, H(t).$$

The function v satisfies boundary condition (17), so boundary condition (3) is also satisfied. □

In this way the inverse Laplace transform of solution (11) is reduced to the evaluation of the function $W(t;\tau)$.

Problem (19)-(22) coincides with the simple one-dimensional problem of propagation of transient shear waves in an initially undisturbed, homogeneous, isotropic viscoelastic half-space by the application of a spatially uniform surface shear stress $H(t)$, if we denote $\tau = x/c$, where x is the coordinate perpendicular to the surface of half-space and c is the wave speed. Formula (23) connects the solutions of considering problems for a viscoelastic and elastic cone with a bit difference on the boundary functions. The right-hand side of this formula is the generalized convolution of the functions v and W. The function v does not depend on the viscous properties of the material, but the effect of all viscous properties of the material to the solution is concentrated on the function W. It is interesting that the elastic and viscoelastic properties of the material are separated and influence each other under the generalized convolution law. Moreover, the function W is universal and may be used for all transient dynamical problems for linear viscoelastic materials.

4. The function $W(t;\tau)$

A reasonably natural way of solving problem (19)-(22) is to apply the Laplace transform. Using the Laplace transform and taking the relation $s\bar{\mu} = \mu_0/\left(1+\bar{G}\right)$ between the Laplace transform $\bar{G}$ of the greep kernel $G(t)$ and the Laplace transform of the relaxation function $\bar{\mu}$ into account, we find $\bar{W}(s;\tau) = \frac{1}{s}\exp\left[-\tau s\sqrt{1+\bar{G}}\right]$.

The function $\bar{W}(s;\tau)$ is analytic on the right half-plane and we can write the solution $W(t;\tau)$ through the inverse Laplace transform

$$W(t;\tau) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty}\frac{1}{s}e^{st-\tau s\sqrt{1+\bar{G}(s)}}ds, \quad p>0. \tag{24}$$

The evaluation of the integral in (24) for the real viscoelastic material is practically very important. But it is not an easy matter and all examples available in the literature refer to the simplest $G(t)$, which correspond to simple relations between strain and stress [1, 7, 10, 14, 19]. The contour integral used here becomes very difficult even in case of the smallest complications of $\bar{G}(s)$ on s. Therefore the method of contour integration becomes unfit for more real relations between stresses and strains. Though integral (24) may be reduced to the real generalized integral, involving the real and imaginary parts of the Laplace transform of the relaxation function, this approach is not useful for both practical and theoretical applications. Here we will use the method of expansion of the function $\bar{W}(s;\tau)$ for the absolutely and uniformly convergent series on the powers of $\bar{G}(s)$.

Theorem 4.1. *If $\bar{G}(s)$ is analytic in the domain $Res \geq p > 0$ and for some positive ω $\left|s^2\bar{G}/\left(\omega^2+s^2\right)\right| < 1$, then the function $\bar{W}(s;\tau)$ is represented as the absolutely and uniformly convergent series on the power of $\bar{G}(s)$*

$$\bar{W} = \frac{1}{s}e^{-\tau s} + \sqrt{\frac{2\tau}{\pi s}}\sum_{n=1}^{\infty}\frac{(-\tau)^n}{2^n n!}\left(s\bar{G}\right)^n K_{n-1/2}(s\tau), \tag{25}$$

where $K_{n-1/2}$ is the modified Bessel function of the second kind.

Proof. Let us represent the function $\bar{W} = \frac{1}{s}\exp\left[-\tau s\sqrt{1+\bar{G}}\right]$ by the integral

$$\bar{W} = \frac{2}{\pi s} \int_0^\infty \frac{\omega \sin(\omega\tau)\, d\omega}{\omega^2 + s^2 \left(1 + \bar{G}\right)} = \frac{2}{\pi s} \int_0^\infty \frac{\omega}{\omega^2 + s^2} \frac{1}{1 + \frac{s^2 \bar{G}}{\omega^2 + s^2}} \sin(\omega\tau)\, d\omega.$$

In the right half-plane $Res \geq p$ we have $\left| s^2 \bar{G} / \left(\omega^2 + s^2\right) \right| < 1$, so we may expand the integrand to the geometric series

$$\bar{W} = \frac{2}{\pi s} \int_0^\infty \sum_{n=1}^{\infty} \frac{\left(-s^2 \bar{G}\right)^n}{\left(\omega^2 + s^2\right)^{n+1}} \omega \sin(\omega\tau)\, d\omega.$$

As the series is absolutely and uniformly convergent and the terms of series are continuous functions of ω, the evaluation of the integral term by term gives formula (25). Moreover, using the asymptotic expansion of the function $K_{n-1/2}(\tau s)$ for large order [4], we see that each term of series in (25) is numerically less than the corresponding term of the series of the function $\exp\left(\tau \left|s\bar{G}\right| / 2\right)$. So series (25) is absolutely and uniformly convergent for all finite τ and its sum tends to zero for $\tau \to \infty$.

Let us estimate the remainder term R_{n+1} of series (25) which may be written as follows

$$R_{n+1} = \sqrt{\frac{2\tau}{\pi s}} \sum_{m=0}^{\infty} \frac{\left(-\tau s \bar{G}\right)^{n+1+m}}{2^{n+1+m} (n+1+m)!} K_{n+m+1/2}(s\tau).$$

Using the boundedness of the function $K_\nu(\tau s)$ for $|s| \geq p > 0$ and the inequality $(n+1+m)! \geq (n+1)!m!$, it is easy to find

$$|R_{n+1}| \leq \frac{\left(\tau \left|s\bar{G}\right|\right)^{n+1} M}{2^{n+1} (n+1)!} e^{\tau \left|s\bar{G}\right|/2},$$

where we denote $\left|\sqrt{2\tau/\pi s} K_{n+m+1/2}(\tau s)\right| < M$. We see that the remainder term R_{n+1} is proportional to $\bar{G}^{n+1}$ and rapidly tends to zero while n gets bigger. □

From the tables of the inverse Laplace transforms we find

$$s^{-1/2} e^{\tau s} K_{n-1/2}(\tau s) \div \sqrt{\frac{\pi}{2\tau}} P_{n-1}(1 + t/\tau),$$

where $P_{n-1}(z)$ is the Legendre polinom [19]. Using this formula we find from (25)

$$W(t;\tau) = H(t-\tau) \left[1 + \sum_{n=1}^{\infty} \frac{(-\tau)^n}{2^n n!} \int_\tau^t G_n^{(n)}(t-\varsigma) P_{n-1}(\varsigma/\tau)\, d\varsigma\right], \tag{26}$$

where $\bar{G}^n(s) \div G_n(t)$ are the iterated kernels which may be obtained as

$$G_1(t) \equiv G(t), \quad G_n(t) = \int_0^t G_1(t-\varsigma) G_{n-1}(\varsigma)\, d\varsigma, \quad n = 2, 3, ...$$

and $G_n^{(n)}(t) = d^n G_n(t)/dt^n$. This formula shows how the creep kernel affects the wave behavior in viscoelastic material.

It is seen from (26) that the velocity of wave propagation determined by the instantaneous modulus and discontinuity can occur at τ on the wave front at time $t = \tau$. The first term in the bracket is the solution of the corresponding elastic problem and all the terms under the summation sign are related to the viscous property of medium. It is seen how the creep kernel affects the wave behavior. Let us investigate the formula near the wave front for which

$0 < t-\tau << 1$. Using $P_{n-1}(\varsigma/\tau) \approx P_{n-1}(1) = 1$ and the easily verified equality $G_n^{(n-1)}(0) = G_0^n$, from (26) we get

$$W(t;\tau) \approx e^{-\tau G_0/2} H(t-\tau).$$

This is the expression of discontinuity induced by a one-dimensional wave with the initial condition (16). It follows from this formula that unlike an elastic medium, where the amplitude of discontinuity remains unchanged, in a viscoelastic medium the amplitude of discontinuity decreases exponentially in time. According to this formula, the rate of decrease in the amplitude of discontinuity is determined by the initial value of the creep kernel.

Using the series expression of the function $K_{n-1/2}$ we find

$$\bar{W} = \frac{1}{s} e^{-\tau s} + \frac{1}{s} e^{-\tau s} \sum_{n=1}^{\infty} \frac{\left(-\bar{G}\right)^n}{2^n n!} \sum_{m=0}^{n-1} \frac{(2n-m-2)!}{m!\,(n-m-1)!} (2s\tau)^{m+1}$$

from (25). The inverse Laplace transform of this function gives the another representation of $W(t;\tau)$

$$W(t;\tau) = H(t-\tau)\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} n!} \sum_{m=0}^{n-1} \frac{(2n-m-2)!}{m!\,(n-m-1)!} (2\tau)^{m+1} G_n^{(m)}(t-\tau)\right].$$

This approach is obtained and investigated in [15, 21]. But the series in (26) more quickly convergent and from (26) it is seen the "near exponentially" damping characters of the solution on the wave front.

Theorem 4.2. *The function* $W(t;\tau)$ *represented by formula* (26) *is the exact solution of the problem* (19)-(22).

Proof. By direct substitution it is not difficult to verify that the function $W(t;\tau)$ satisfies the equation

$$\frac{\partial^2 W}{\partial \tau^2} - \frac{\partial^2 W}{\partial t^2} - \int_0^t G(t-\varsigma) \frac{\partial^2 W(\varsigma;\tau)}{\partial \varsigma^2} d\varsigma = 0$$

which is the other expression of equation (19) written by using creep kernel.

Satisfactions of conditions (20)-(22) are apparently seen, as far as all the terms of (26) are multiplied by the Heaviside function $H(t-\tau)$. □

Regular kernel. Let the kernel $G(t)$ be represented in the form $G(t) = G_0 - \Phi(t)$, where $G_0 = G(0)$, $\Phi(0) = 0$. Then we find $\bar{W} = \frac{1}{s}\exp\left(-\tau\sqrt{\alpha^2-\theta}\right)$, where $\alpha^2 = s^2 + sG_0$, $\theta = s^2\bar{\Phi}$. Like (25), we get

$$\bar{W} = \frac{1}{s} e^{-\tau\alpha} + \frac{1}{s}\sqrt{\frac{2\tau\alpha}{\pi}} \sum_{n=1}^{\infty} \frac{\tau^n}{2^n n!} \left(\frac{\theta}{\alpha}\right)^n K_{n-1/2}(\alpha\tau). \tag{27}$$

We will make the next notations to find the original

$$\tau^n \sqrt{\frac{2}{\pi}} \alpha^{-n+1/2} K_{n-1/2}(\alpha\tau) \div \left(\frac{G_0}{2}\right)^{1-n} e^{-\frac{tG_0}{2}} \left(t^2-\tau^2\right)^{\frac{n-1}{2}} I_{n-1}\left(\frac{G_0}{2}\sqrt{t^2-\tau^2}\right) \equiv F_n(t,\tau),$$

$$\Pi_1(t) = \Phi'(t),\ \Pi_n(t) = \Pi_1(0)\,\Pi_{n-1}(t) + \int_0^t \Pi_{n-1}(t-\varsigma)\, d\Pi_1(\tau),$$

$$\int_{\tau}^{t} \Pi_n (t-\varsigma) F_n (\tau,\varsigma) d\varsigma = \varphi_n (\tau,t),$$

where I_n is the modified Bessel function of the first kind. Then the inverse Laplace transform of the function (27) is expressed as follows

$$W(t;\tau) = H(t-\tau)\left[e^{-\frac{\tau G_0}{2}} + \frac{\tau G_0}{2}\int_{\tau}^{t} e^{-\frac{\varsigma G_0}{2}} \frac{I_1\left(\frac{G_0}{2}\sqrt{\varsigma^2-\tau^2}\right)}{\sqrt{\varsigma^2-\tau^2}} d\varsigma + \sum_{n=1}^{\infty}\frac{\tau}{2^n n!}\varphi_n(t,\tau)\right]. \quad (28)$$

Here the first two terms correspond to the Maxwell model which is obtained for $\Phi(t) \equiv 0$, i.e. for Maxwell model $G(t) = G_0 H(t)$.

Theorem 4.3. *The function $W(t;\tau)$ represented by formula (28) is the exact solution of the problem (19)-(22) for the creep kernel $G(t) = G_0 - \Phi(t)$, where $\Phi(t)$ is a monotonically increasing continuous function so that $\Phi(0) = 0$, $\Phi(\infty) = G_0$ and $G_0 = G(0)$. The series in (28) is absolutely and uniformly convergent for any finitet.*

Proof. Satisfactions of conditions (20)-(22) are apparently seen. By direct substitution it is not difficult to verify that the function (28) satisfies equation (19), too. Let us prove the convergence of the series in (28). The kernel $G(t)$ is monotonically decreasing concave function of time, so the function $\Pi_1(t)$ is non-negative, bounded on $t = 0$ and monotonically decreasing up to zero. Consequently,

$$|\Pi_2(t)| < \Pi_1^2(0) + \int_0^t |\Pi_1(t-\varsigma)\, d\Pi_1(\varsigma)| < \Pi_1^2(0) + \Pi_1(0)\int_0^t |d\Pi_1(\varsigma)| < 3\Pi_1^2(0), \ldots,$$

$$|\Pi_n(t)| < 3^{n-1}\Pi_1^n(0).$$

Since the Bessel function $I_n(x)$ is monotonically increasing, we get

$$|F_n| < \left(\frac{G_0}{2}\right)^{1-n} t^{n-1} I_{n-1}\left(\frac{G_0 t}{2}\right)$$

and using the integral representation of $I_n(x)$ we find

$$I_{m-1}\left(\frac{G_0 t}{2}\right) = \frac{\left(\frac{G_0 t}{2}\right)^{n-1}}{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\cosh\left(\frac{G_0 t}{2}\cos\varsigma\right)\sin^{2(n-1)}\varsigma d\varsigma < \frac{(n-1)!\,(G_0 t)^{n-1}}{(2n-1)!}e^{\frac{G_0 t}{2}}.$$

Then we have estimation

$$|F_m| < \frac{2^{n-1}(n-1)!}{(2n-1)!}t^{2(n-1)}.$$

Using the estimations for the functions Π_n and F_n we estimate φ_m :

$$|\varphi_n| < \frac{(n-1)!\left(6\Pi_1(0)\,t^2\right)^n}{3\,(2n-1)!\,(2n-1)\,t}.$$

Then the series in (28) will have the next majorant

$$\sum_{n=1}^{\infty}\frac{\left(t\sqrt{3\Pi_1(0)}\right)^{2n-1}}{(2n-1)!n\,(2n-1)}$$

which is convergent for all $t \geq 0$. So the series in (28) is absolutely and uniformly convergent for all t. $\square$

Example 4.1. Let $G(t) = \sum_{i=1}^{N} A_i e^{-\delta_i t}$, where $A_i > 0$ and $\delta_i > 0$ are constant. Then

$$\bar{W} = \frac{1}{s} \exp\left[-\tau\sqrt{(s+\eta)^2 - \xi^2 + \sum_{i=1}^{N} A_i \delta_i^2 (s+\delta_i)^{-1}}\right],$$

where

$$\alpha = (s+\eta)^2 - \xi, \quad 2\eta = \sum_{i=1}^{N} A_i, \quad \xi^2 = \sum_{i=1}^{N} A_i \delta_i + \eta^2.$$

Similar to (27), we get

$$\bar{W} = \frac{1}{s} e^{-\tau\alpha} + \frac{1}{s}\sqrt{\frac{2\alpha\tau}{\pi}} \sum_{n=1}^{\infty} \frac{1}{2^{2n} n!} \left(-\sum_{i=1}^{N} \frac{A_i \delta_i^2}{s+\delta_i}\right)^n \alpha^{-n} K_{n-1/2}(\alpha\tau).$$

Here we find

$$W(t;\tau) = \sum_{n=0}^{\infty} q_n(t;\tau),$$

where the first two terms have been easily got

$$q_0(t;\tau) = e^{-\eta\tau} H(t-\tau) + \tau\xi \int_\tau^t e^{-\eta\varsigma} \frac{I_1\left(\xi\sqrt{\varsigma^2-\tau^2}\right)}{\sqrt{\varsigma^2-\tau^2}} d\varsigma,$$

$$q_1(t;\tau) = -\frac{c\tau}{2} \sum_{i=1}^{N} A_i \delta_i \int_\tau^t e^{-\eta\varsigma} \left[1 - e^{-\delta_i(t-\varsigma)}\right] I_0\left(\xi\sqrt{\varsigma^2-\tau^2}\right) d\varsigma.$$

The next functions $q_n(t;\tau) \quad (n \geq 2)$ may be obtained as follows. Let us denote

$$\bar{T}_n = \left(\sum_{i=1}^{N} \frac{A_i \delta_i^2}{s+\delta_i}\right)^n$$

and

$$T_1(t) = \sum_{i=1}^{N} A_i \delta_i^2 e^{-(\delta_i - \eta)t}, \quad T_n(t) = \int_0^t T_1(t-\varsigma) T_{n-1}(\varsigma) d\varsigma, \quad R_n(t) = \int_0^t T_n(\varsigma) d\varsigma,$$

then $q_n(t;\tau)$ is obtained as

$$q_n(t;\tau) = \frac{(-1)^n}{2^{2n} n!} \sum_{m=0}^{n-1} \frac{(2n-m-2)!}{m!(n-m-1)!} \frac{(2\tau)^{m+1}}{\Gamma(n-m/2)} \int_0^t R_n(t-\varsigma) e^{-\eta\varsigma} \times$$
$$\times \int_\tau^\varsigma I_0\left(\xi\sqrt{\varsigma^2-v^2}\right) (\varsigma - v - \tau)^{n-1-m/2} dv d\varsigma.$$

Example 4.2. For $N = 1$, which corresponds to the standard linear solid, we have $2\eta = A$,$\xi^2 = A\delta + \eta^2$. Then $q_0(t;\tau)$ and $q_n(t;\tau)$ remain the same as above with

$$T_n(t) = \left(A\delta^2\right)^n e^{-\delta t} t^{n-1} / (n-1)!$$

and

$$q_1(t;\tau) = -\frac{c\tau}{2}A\delta \int_\tau^t e^{-\eta\varsigma}\left[1 - e^{-\delta(t-\varsigma)}\right] I_0\left(\xi\sqrt{\varsigma^2 - \tau^2}\right) d\varsigma.$$

Example 4.3. The constitutive equations of viscoelastic materials, obtained by using the fractional derivative models, often appear in literature recently [6, 11]. For this equations creep kernel becomes unbounded at the initial point $t = 0$. Let us consider the simplest unbounded integrable Abel kernel $G(t) = At^{-\delta}, \quad A > 0,\ 0 < \delta < 1$. From (26) we find

$$W(t;\tau) = H(t-\tau)\left\{1 + \sum_{n=1}^{\infty} \frac{[-\tau A\Gamma(1-\delta)]^n}{2^n n!\Gamma(-n\delta)} \int_\tau^t (t-\varsigma)^{-1-n\delta} P_{n-1}(\varsigma/\tau)\, d\varsigma\right\}, \qquad (29)$$

where $\Gamma(z)$ is the Euler gamma function and all terms corresponding to the whole negative δn are missed. In this case the behavior of solution (29) for small positive $t - \tau$ is interesting. Putting $P_{n-1}(\varsigma/\tau) \approx P_{n-1}(1) = 1$ from (29) we find

$$W(t;\tau) \approx \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(1-n\delta)}\left[-\frac{A\tau}{2}\Gamma(1-\delta)(t-\tau)^{-\delta}\right]^n.$$

Using the formula

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C (-\xi)^{-z} e^{-\xi} d\xi,$$

where the contour C comes from the infinity over the positive poly-axis to zero, turns around the origin and goes back to infinity below the positive poly-axis and denotes

$$(-\xi)^{-1} = f(\xi), \quad (t-\tau)^{-\delta} = h, \quad A\tau\Gamma(1-\delta)(-\xi)^{\delta}\Big/2 + \frac{\xi}{h} = S(\xi),$$

we write the last formula in the form

$$W(t;\tau) \approx \frac{i}{2\pi}\int_C f(\xi)\, e^{-hS(\xi)} d\xi.$$

For small $t - \tau$ the parameter h becomes larger, so we may use the saddle-point method to calculate the contour integral. Not reducing the standard formulas of this method we will write the main term of asymptotic expansion

$$\begin{aligned} W(t;\tau) \approx & \frac{1}{\sqrt{2\pi(1-\delta)}}\left[\frac{2}{A\delta\tau\Gamma(1-\delta)}\right]^{\frac{1}{2(1-\delta)}} (t-\tau)^{\frac{\delta}{2(1-\delta)}} \times \\ & \times \exp\left\{-\frac{1-\delta}{\delta}\left[\frac{\delta\tau A\Gamma(1-\delta)}{2}\right]^{\frac{1}{1-\delta}} (t-\tau)^{\frac{-\delta}{1-\delta}}\right\}. \end{aligned} \qquad (30)$$

The formula is correct for $\tau > 0$. It is seen from this formula that $W \to 0$ for $t - \tau \to 0$, i.e. the solution for weakly singular kernel tends to zero on the wave front propagating by the elastic wave speed. In this case the decay of the solution on the wave front is much more quicker in comparison with the case of the regular creep kernel for which $W \to \exp(-\tau G_0/2)$. Equation (30) implies that $W(t;\tau)$ is continuous in the domain $t > 0$ and $\tau > 0$. Therefore, the presence of the weakly singularity in the hereditary kernel leads to smoothing any discontinuity in the boundary and initial data. Moreover, the smoothness of any solution near the wave front increases with the growth of time t or τ (i.e. with the growth of space coordinate).

The function $W(t;\tau)$ obtained above is useful for all values of time. For the viscoelastic material with finite creep the behavior of the solution of dynamic problems for large values of time is of interest. In this case we propose the quicker convergent series for the expression of the function $W(t;\tau)$. The image $\bar{W}(s;\tau)$ may be represented as follows

$$\bar{W}(s;\tau) = \frac{1}{s}\exp\left[-\frac{\tau s}{c_1}\sqrt{\frac{\mu_\infty}{s\bar{\mu}}}\right],$$

where $c_1 = \sqrt{\mu_\infty}$. Denoting $\tau_1 = \tau/c_1$, $\bar{G} = (\mu_\infty - s\bar{\mu})/s\bar{\mu}$ and using formula (25) we get

$$\bar{W}(s;\tau) = \frac{1}{s}e^{-s\tau/c_1} + \sqrt{\frac{2\tau}{\pi s c_1}}\sum_{n=1}^{\infty}\frac{(s\tau/c_1)^n}{2^n n!}\left(\frac{s\bar{\mu}-\mu_\infty}{s\bar{\mu}}\right)^n K_{n-1/2}\left(\frac{s\tau}{c_1}\right). \tag{31}$$

This series is not alternating series and is absolutely and uniformly convergent if the condition $\left|s^2(s\bar{\mu}-\mu_\infty)/\left(s^2+\omega^2\right)s\bar{\mu}\right| < 1$ is satisfied for some positive ω. The left-hand side of the inequality tends to zero for $s \to 0$ and has the greatest value $\varepsilon = (\mu_0 - \mu_\infty)/\mu_0$ for $s \to \infty$, that is the inequality is satisfied for all values of the parameter s. The remainder term of the series (31) has the estimate

$$|R_{n+1}| < \frac{1}{|s|}\left|\frac{s\bar{\mu}-\mu_\infty}{s\bar{\mu}}\right|^{n+1}\left|1 - e^{-s\tau/c_1}\frac{F_n(s\tau/c_1)}{2^n n!}\right|,$$

where the functions $F_n(z)$ are defined from the recursion relation

$$F_0(z) = 1, \quad F_n(z) = (z+2n)F_{n-1}(z) - zF'_{n-1}(z).$$

Here we see quick convergence of series (31) for small values of parameter$s$.

Denoting

$$M_1(t) = \varepsilon\mu_0\Pi'(t) - G(t), \quad M_n(t) = \int_0^t M_1(t-\varsigma)M_{n-1}(\varsigma)\,d\varsigma$$

we obtain the inverse Laplace transform of (31)

$$W(t;\tau) = H(t-\tau)\left[1 + \sum_{n=1}^{\infty}\frac{(\tau/c_1)^n}{2^{2n}n!}\int_{\tau/c_1}^{t} M_n^{(n)}(t-\varsigma)P_{n-1}(\varsigma c_1/\tau)\,d\varsigma\right], \tag{32}$$

where $\Pi(t)$ is the creep function. As for $t - \tau/c_1 \to \infty$ the functions $M_n^{(n)}(t-\tau/c_1) \to 0$, we have $W(t;\tau) = H(t-\tau/c_1)$. It denotes that for the large values of time viscoelastic material behaves as an elastic one with the equilibrium elastic modulus.

5. Solution for the viscoelastic cone

If we want to know the inversion of the function $\bar{W}_{in}(s;\tau) = \bar{a}_{in}(s)\exp[-\tau c\eta(s)]$ then

$$W_{in}(t;\tau) = \int_0^t W(t-\varsigma;\tau)\,da_{in}(\varsigma)$$

will be written. Using theorem1, from (11) we find the solution of the problem considered for the viscoelastic cone for given functions $a_i(\theta,t) \quad (i=1,2)$ in the form

$$u(r,\theta,t) = \sum_{n=1}^{\infty}P^1_{\alpha_n}(x)\left[\left(\frac{r_1}{r}\right)^{\alpha_n+1}\frac{r_2^{2\alpha_n+1}-r^{2\alpha_n+1}}{r_2^{2\alpha_n+1}-r_1^{2\alpha_n+1}}a_{1n}(t) + \left(\frac{r_2}{r}\right)^{\alpha_n+1}\frac{r^{2\alpha_n+1}-r_1^{2\alpha_n+1}}{r_2^{2\alpha_n+1}-r_1^{2\alpha_n+1}}a_{2n}(t)\right] -$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}P^1_{\alpha_n}(x)\frac{\pi\lambda_{kn}J_{\alpha_n+1/2}(r_1\lambda_{kn})J_{\alpha_n+1/2}(r_2\lambda_{kn})}{J^2_{\alpha_n+1/2}(r_1\lambda_{kn})-J^2_{\alpha_n+1/2}(r_2\lambda_{kn})}\left[\sqrt{\frac{r_1}{r}}B_{nk}(r)\int_0^t W_{1n}(t;\tau)\sin(\lambda_{kn}c\tau)\,d\tau-\right.$$

$$\left.-\sqrt{\frac{r_2}{r}}\tilde{B}_{nk}(r)\int_0^t W_{2n}(t;\tau)\sin(\lambda_{kn}c\tau)\,d\tau\right], \tag{33}$$

where $\tilde{B}_{nk}(r)$ is obtained from $B_{nk}(r)$ by replacing $r_2 \to r_1$.

We may summarize the results obtained by the following way.

Theorem 5.1. *Let the functions $a_i(\theta,t)$ $(i=1,2)$ be twice continuous differentiable with respect to θ on the interval $(0,\theta_0)$ and piecewise continuous with respect to t on any finite interval on the axis $t>0$. Moreover, let $a_i(\theta,t)$ are zero for $t<0$ and can not grow faster than the exponential function with respect to t and $a_i(0,t)=0$. Then the function $u(r,\theta,t)$ defined by formula* (32) *is the solution of initial-boundary-value problem* (1)-(4) *for any relaxation function $\mu(t)$. The solution exists for all time and depends continuously on the data.*

The stress $\sigma_{r\varphi}$ is represented by the formula

$$\sigma_{r\varphi}=G^*\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)=\sum_{n=1}^{\infty}\sigma_{r\varphi n},\quad \sigma_{r\varphi n}=P^1_{\alpha_n}(x)\left\{a_{1n}(t)*\left(\frac{d}{dr}-\frac{1}{r}\right)\left[\sqrt{\frac{r_1}{r}}\Omega_{1n}(r,t)\right]+\right.$$

$$\left.+a_{2n}(t)*\left(\frac{d}{dr}-\frac{1}{r}\right)\left[\sqrt{\frac{r_2}{r}}\Omega_{2n}(r,t)\right]\right\}. \tag{34}$$

The torsion moment is

$$M=-2\pi r^3\int_1^{x_0}\sqrt{1-x^2}\sigma_{r\varphi}dx.$$

Using formula $P^1_1(x)=-\sqrt{1-x^2}$ $(\alpha_1=1)$ and the ortogonality of the Legendre functions on the domain $[x_0,1]$, we come into conclusion that only the first term in formula (34) form the non zero torsion moment. Thus solution (33) consists of sum of the self-balanced $(\alpha_n\neq 1)$ and non self-balanced $(\alpha_1=1)$ terms. The non self-balanced part of the solution coincides with the elementary solution for which the spherical sections $r=const$ turns safely around the axis of the cone and the arches $r\theta$ do not bend on this surface. In the section corresponding to the self-balanced terms such as arches bend and the number of bending increases depending on n. This is the case that for $\alpha_n\to\infty$ the number of zeros of the function $P^1_{\alpha_n}(x)$ increases on the segment $[x_0,1]$.

For the small values of time for elastic cone we obtain

$$u\approx\sum_{n=0}^{\infty}\left\{\frac{r_1}{r}\left[a_1(\theta,t-y_{1n}/c)-a_1(\theta,t-y_{2n}/c)\right]+\right.$$

$$\left.+\frac{r_2}{r}\left[a_2(\theta,t+z_{1n}/c)-a_2(\theta,t+z_{2n}/c)\right]\right\}, \tag{35}$$

where

$y_{in}=(2n+1)(r_2-r_1)+(-1)^i(r_2-r)$,

$z_{in}=(2n+1)(r_2-r_1)+(-1)^i(r_1-r)$ $\quad(i=1,2)$.

As we see, the spherical sections $r=const$ turn as a boundary sphere by the amplitude proportionally to r^{-1}.

For the viscoelastic cone with the regular creep kernel from (35) and (28) we find

$$u \approx \sum_{n=0}^{\infty} \left\{ \frac{r_1}{r} \left[a_1\left(\theta, t - y_{1n}/c\right) e^{-G_0 y_{1n}/2c} - a_1\left(\theta, t - y_{2n}/c\right) e^{-G_0 y_{2n}/2c} \right] + \right.$$
$$\left. + \frac{r_2}{r} \left[a_2\left(\theta, t + z_{1n}/c\right) e^{-G_0 z_{1n}/2c} - a_2\left(\theta, t + z_{2n}/c\right) e^{-G_0 z_{2n}/2c} \right] \right\}$$

for the small values of time. The displacement is decreased as exponential functions on the wave front.

If we replace the sum in (11) by the contour integral, put $a_{2n} = 0$ and then take the limit for $r_2 \to \infty$, we get the solution for the semi-infinite cone. In this case

$$\Omega_{1n}\left(r, t\right) = \left(\frac{r_1}{r}\right)^{\alpha_n + 1/2} \delta\left(t\right) - \sum_{k=1}^{\infty} \frac{c Y_{\alpha_n + 1/2}\left(r \omega_{kn}\right)}{r_1 Y'_{\alpha_n + 1/2}\left(r_1 \omega_{kn}\right)} \sin\left(\omega_{kn} c t\right),$$

where ω_{kn} are all zeros of the function $Y_{\alpha_n + 1/2}\left(r_1 \omega\right)$ and

$$u\left(r, \theta, t\right) = \sum_{n=1}^{\infty} P^1_{\alpha_n}\left(x\right) \left[\left(\frac{r_1}{r}\right)^{\alpha_n + 1} a_{1n}\left(t\right) - \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{c Y_{\alpha_n + 1/2}\left(r \omega_{kn}\right)}{r_1 Y'_{\alpha_n + 1/2}\left(r_1 \omega_{kn}\right)} \sqrt{\frac{r_1}{r}} \int_0^t W_{1n}\left(t; \tau\right) \sin\left(c \omega_{kn} \tau\right) d\tau \right]. \tag{36}$$

6. Half-sphere and half-space with the half-spherical cavity

If we put $\theta_0 = \pi/2$ in (33) we get the solution for the half-sphere with thickness $h = r_2 - r_1$. In this case formula (36) gives us the solution for the half-space. Equation (8) returns to $P^2_\alpha\left(0\right) = 0$ and has the roots $\alpha_n = 2n - 1, \quad n = 0, 1, 2, ...$ The Bessel functions of the subscript $\alpha_n + 1/2 = 2n - 1/2$ return to elementary functions and the zeros of the function $Y_{\alpha_n + 1/2}\left(r_1 \omega\right)$ situated symmetrically around the origin $\omega = 0$. The form of formula (33) does not change, but formula (36) becomes as

$$u\left(r, \theta, t\right) = \sum_{n=1}^{\infty} P^1_{2n-1}\left(x\right) \left[\left(\frac{r_1}{r}\right)^{2n} a_{1n}\left(t\right) - \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{2c Y_{2n - 1/2}\left(r \omega_{kn}\right)}{r_1 Y'_{2n - 1/2}\left(r_1 \omega_{kn}\right)} \sqrt{\frac{r_1}{r}} \int_0^t W_{1n}\left(t; \tau\right) \sin\left(c \omega_{kn} \tau\right) d\tau \right]. \tag{37}$$

Here ω_{kn} are the positive roots of the function $Y_{2n-1/2}\left(r_1 \omega\right)$.

7. Half-space

If $r_1 \to 0$ and $2\pi r_1^2 a_1\left(t\right) \to a^*\left(t\right)$ from (37) we get

$$u\left(r, \theta, t\right) = -\frac{\sin\theta}{2\pi} \frac{\partial}{\partial r} \left[\frac{a^*\left(t - r/c\right)}{r} \right]$$

for the elastic and

$$u\left(r, \theta, t\right) = -\frac{\sin\theta}{2\pi} \frac{\partial}{\partial r} \left[\frac{1}{r} \int_{r/c}^t W\left(\tau; r/c\right) d a^*\left(t - \tau\right) \right]$$

for the viscoelastic half-space.

As we see, receding from the center the displacement decreases proportionally to r^{-2}. The factor $\sin\theta$ shows that the displacement of the free surface points is greater with respect to the displacement of other points situated on the sphere $r = const$.

8. Cylinder

Consider the case $\theta_0 \to 0, \quad r_1 \to \infty, \quad \theta_0 r_1 \to r_0 = const > 0$. Equation (8) turns to

$$(\alpha+2)(\alpha+1)\alpha(\alpha-1)\left(\alpha+\frac{1}{2}\right)^{-2} J_2\left((2\alpha+1)\sin\theta_0/2\right) = 0$$

and has the roots

$$\alpha = -2, \quad \alpha = -1, \quad \alpha = 0, \quad \alpha = 1, \quad (2\alpha+1)\sin\theta_0/2 = r_0\rho_n, \quad n = 1, 2, ...,$$

where ρ_n are the roots of the equation $J_2(r_0\rho) = 0$. Here the first three roots do not give the displacement and $\alpha = 1$ corresponds to the elementary solution, when the cylinder turns as absolutely rigid body. In the last equation for $\theta_0 \to 0$ the roots α_n become infinitely large. Using the formula $P_\alpha^1(x) \approx 2\alpha(\alpha+1) J_1(\rho_n\zeta)/(2\alpha+1)$, we get the Fourier-Bessel coefficients of the functions $\bar{a}_i(\zeta, s)$

$$\bar{a}_{in} = \frac{1}{r_0^2 J_1^2(r_0\rho_n)} \int_0^{r_0} \bar{a}_i(\zeta, s) J_1(\rho_n\zeta)\zeta d\zeta, \quad \bar{a}_{i0} = \int_0^{r_0} \bar{a}_i(\zeta, s)\zeta^2 d\zeta$$

from formula (10). Here r_0 is the radius of the cylinder and ζ is the radial coordinate.

Using these expressions and the asymptotic expansions of the Bessel functions for the large order and large variable [4], we get

$$\bar{u} = \bar{a}_{10} R e^{-z\eta} + \sum_{n=1}^{\infty} \bar{a}_{1n} J_1(\rho_n R) e^{-z\sqrt{\eta^2+\rho_n^2}}$$

from formula (11), where $z = r - r_1$ and R are the cylindrical coordinates. This is the well-known image of the solution of the problem considered for the cylindrical bars in [18].

9. Contribution to operational calculus

The main contribution to operational calculus may be expressed as follows

Theorem 9.1. *Let the image $\bar{f}(s)$ of the function $f(t)$ be given and the analytic function $\bar{G}(s)$ be so that the inverse Laplace transform of the function $\frac{1}{s}\exp\left[-\tau s\sqrt{1+\bar{G}(s)}\right]$ is the function $W(t;\tau)$ represented by one of the formulas* (26), (30) *or* (32). *Then inverse Laplace transform of the function $\frac{1}{s}f\left(s\sqrt{1+\bar{G}(s)}\right)$ is*

$$\int_0^t f(\tau) W(t;\tau) d\tau.$$

Proof. Using the property $W(t;\tau) = 0$ for $t \leq \tau$ and calculating the Laplace transform of $\int_0^t f(\tau) W(t;\tau) d\tau$ we find

$$\int_0^\infty e^{-st} \int_0^t f(\tau) W(t;\tau) d\tau dt = \int_0^\infty e^{-st} \int_0^\infty f(\tau) W(t;\tau) d\tau dt =$$
$$= \int_0^\infty f(\tau) \int_0^\infty e^{-st} W(t;\tau) dt d\tau =$$

$$= \frac{1}{s}\int_0^\infty f(\tau)\, e^{-\tau s\sqrt{1+\bar{G}(s)}} d\tau = \frac{1}{s}\bar{f}\left(s\sqrt{1+\bar{G}(s)}\right).$$

Note that such a theorem exist in [12] and for simplest $\bar{G}(s)$ have in [25], but the function $W(t;\tau)$ for any analytic $\bar{G}(s)$ is constructed here for the first time. □

10. Conclusion

The problem of dynamical torsion of a viscoelastic cone is investigated in the current work. The solution is obtained in the form of generalized convolution of corresponding elastic solution by the solution of auxiliary problem, which corresponds to the one-dimensional problem of propagation of transient shear wave in viscoelastic half-space with any hereditary property. The solution of the auxiliary problem obtained here does not depend on problems considered and may be used to solve all transient dynamical problems for linear viscoelasticity. So the solution of any viscoelastic problem is reduced to the solution of corresponding elastic ones. The formula connecting these solutions may be called correspondence principle of the solution of elastic and viscoelastic dynamical problems. The solution of the auxiliary problem includes all the hereditary property of viscoelastic material and we call it the hereditary factor function.

All problems considered are solved by using the Laplace integral transform. The inverse Laplace transform for an elastic cone is obtained by the method of contour integration. The Laplace transform of the solution of the auxiliary problem is expanded to the absolutely and uniformly convergent series on the powers of the Laplace transform of the creep kernel. So, using convolution theorem, it becomes possible to obtain the inverse transform of considering dynamical problems for any hereditary property of material. The regular kernel, the kernel sum of the exponential functions and weakly singular kernel have been considered in the study. On the wave front propagating by the elastic wave speed the solution decays by the exponential law for a regular kernel, but for weakly singular kernel this decay takes place more quickly. The main energy transported by the wave is concentrated on the wave front propagating by the equilibrium speed $\sqrt{\mu_\infty/\rho}$, for which the decay is independently absent on the kernel.

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